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Analytical Gradients for Gravity Assist Trajectories Using Constant Specific Impulse Engines

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A procedure for calculating the analytical derivatives required to optimize long duration constant specific impulse finite burns and multiple gravity assist trajectories is presented. The analytical derivatives are calculated using the state transition matrix associated with the complete set of the Euler–Lagrange equations of the optimal control problem on each trajectory segment. Another transition matrix maps perturbations across any discontinuities in the state due to a zero sphere of influence patched conic flyby or discontinuities in the equations of motion that occur when the engine turns on or off. As applications, the method is used to find optimal Earth to Saturn trajectories. The state transition matrix derivatives are shown to find optimal trajectories from sets of initial conditions where finite difference derivatives fail to converge.

Nomenclature

C	=	constraint function
c	=	specific impulse
$\boldsymbol{e}$	=	unit vector for adjoint control transformation
$\boldsymbol{g}$	=	gravitational acceleration acting on spacecraft
H	=	Hamiltonian function
$\boldsymbol{h}$	=	orbital angular momentum of spacecraft
$\boldsymbol{I}$	=	identity matrix
J	=	cost function
m	=	spacecraft mass
q	=	number of gravity assists
$\boldsymbol{R}$	=	transformation matrix for adjoint control transformation
$\boldsymbol{R}_{\mathrm{fb}}$	=	transformation matrix used to express the flyby $\Delta \boldsymbol{v}$ in an inertial frame
$\boldsymbol{r}$	=	spacecraft position
r_m	=	periapse on hyperbolic flyby trajectory
S	=	switching function
T	=	thrust magnitude
t	=	time
$\boldsymbol{u}$	=	thrust direction unit vector
$\boldsymbol{v}$	=	velocity
v_∞	=	velocity of spacecraft relative to flyby planet
$\boldsymbol{X}$	=	augmented spacecraft state (position, velocity, mass, and costates)
$\boldsymbol{x}$	=	perturbation of true state from nominal state
y	=	position
α	=	adjoint control parameter
α'	=	adjoint control parameter
β	=	flyby angle
γ, γ'	=	adjoint control parameter
δ	=	time-free variation
δ_{fb}	=	flyby turn angle
$\tilde{\delta}$	=	time-fixed variation
ζ	=	relaxation parameter
κ	=	parameter that affects spacecraft state
$\lambda_m, \boldsymbol{\lambda}_r, \boldsymbol{\lambda}_v$	=	spacecraft mass, position, and velocity costate
$\lambda^{\perp}_{v_{\mathrm{fb}-}}$	=	component of velocity costate at $t_{\mathrm{fb}-}$ that is perpendicular to $v_{\infty i}$
$\lambda^{\perp}_{v_{\mathrm{fb}+}}$	=	component of velocity costate at $t_{\mathrm{fb}+}$ that is perpendicular to $v_{\infty o}$
$\lambda^{\parallel}_{v_{\mathrm{fb}-}}$	=	component of velocity costate at $t_{\mathrm{fb}-}$ that is parallel to $v_{\infty i}$
$\lambda^{\parallel}_{v_{\mathrm{fb}+}}$	=	component of velocity costate at $t_{\mathrm{fb}+}$ that is parallel to $v_{\infty o}$
μ_{fb}	=	gravitational parameter of gravity assist planet
ρ, υ	=	Lagrange multiplier
$\boldsymbol{\Phi}$	=	state transition matrix for spacecraft state described by $\boldsymbol{X}$

Subscripts

f	=	final
fb	=	flyby
i	=	inbound
max	=	maximum
min	=	minimum
o	=	outbound
p	=	flyby planet
s	=	switching
t	=	target planet
0	=	initial
$-$	=	immediately before
$+$	=	immediately after

Superscripts

$*$	=	nominal
$-$	=	perturbed backward
$+$	=	perturbed forward
&	=	partial derivative mapped to a common time

Introduction

MOST techniques to determine optimal low-thrust spacecraft trajectories require derivatives of the cost and/or constraint functions with respect to the free parameters. Usually these derivatives are calculated using finite differences because of the ease of calculating them. Some researchers have used the state transition matrix to calculate derivatives in a limited class of problems (impulsive maneuvers). D'Amario et al.[1] and Sauer[2] use the state transition matrix to calculate partial derivatives that are used to optimize an impulsive $\Delta \boldsymbol{v}$ multiple flyby trajectory. Mirfakhraie and Conway[3] use the state transition matrix to calculate the necessary derivatives for

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a cooperative time-fixed impulsive rendezvous. The state transition matrix in these previous studies only applied to the position and velocity of the spacecraft. Ocampo and Rosborough[4] used the state transition matrix applied to the full spacecraft state and costates to find optimal finite duration thrust trajectories. Zimmer and Ocampo[5] used the state transition matrix applied to the full spacecraft state and costates to find optimal trajectories where the spacecraft state is discontinuous. This paper presents a method for calculating the necessary derivatives for a constant specific impulse (CSI) gravity assist trajectory using the state transition matrix where both the spacecraft state and the spacecraft equations of motion may be discontinuous during the trajectory.

The optimization process in this paper will be accomplished using two different methods to show that the state transition matrix derivatives work in both cases. The first method is an indirect method where calculus of variations is used to derive the optimality conditions resulting in a two-point boundary-value problem (TPBVP). The second method is a hybrid method that uses a continuous control and searches for optimal values of the initial costates. The kinematic boundary conditions are satisfied through the use of equality constraints, and the transversality conditions are satisfied by minimizing the cost function using a sequential quadratic programming code. These methods both require the derivatives of the cost and constraint functions with respect to all of the free parameters. The accuracy with which the optimal trajectory can be found depends on the accuracy of these derivatives. The derivatives calculated using the state transition matrix will be shown to be exact to the accuracy of the integrator used to propagate the equations of motion and the state transition matrix.

This technique of calculating derivatives will be applied to three example problems. The first example is a one-dimensional problem where the acceleration is discontinuous based on the value of the position. This problem allows the analytic gradients to be calculated by an expansion about the nominal motion. These derivatives are then compared to the derivatives calculated using the method presented in this paper to verify that this method gives exact gradients when the equations of motion can be integrated exactly and the state transition matrix is known exactly. The second example is a CSI trajectory that includes a patched conic gravity assist, which is optimized by using the indirect method. In this second example both the spacecraft state and the spacecraft equations of motion are discontinuous. The discontinuities in the spacecraft state are caused by gravity assists, and the discontinuities in the equations of motion are caused by the spacecraft thrust being discontinuous. Because the spacecraft equations of motion and state transition matrix must be integrated numerically, no analytic value for the gradients is available. Both the finite difference derivatives and the analytic derivatives are used to find an optimal interplanetary gravity assist trajectory. A comparison is made on the robustness of convergence to an optimal solution using both methods and the same initial guess for the free parameters. The third example problem is a CSI trajectory that is optimized using the hybrid method. Again, the equations of motion are discontinuous and no analytical derivatives are available. Consequently, the finite difference derivatives and state transition matrix method derivatives are both used to find an optimal trajectory. In this example, state transition matrix derivatives converge to the optimal trajectory that has a different thrusting structure than the initial guess, whereas the central difference derivatives do not.

Calculating Partial Derivatives

Finite Difference Method

One method used to calculate partial derivatives in a trajectory optimization algorithm is to use finite differences. The primary reason to use finite difference derivatives is the ease with which they can be implemented. Finite difference derivatives can be used without modification to the formulas even if the equations of motion and the spacecraft state are discontinuous during trajectory segments. Figure 1 shows the definitions of $\boldsymbol{X}^+$ and $\boldsymbol{X}^-$, as well as a trajectory where the equations of motion are discontinuous at one interior point t_s. Notice that the perturbation to the variable κ causes both the time when the equations of motion are discontinuous as well as the amount of perturbation in the equations of motion to change. Equation (1) provides a second-order method to calculate the finite difference derivatives of the state at time t_1 with respect to a parameter κ that affects the state at t_0. This equation is valid even when the equations of motion and the spacecraft state are discontinuous:

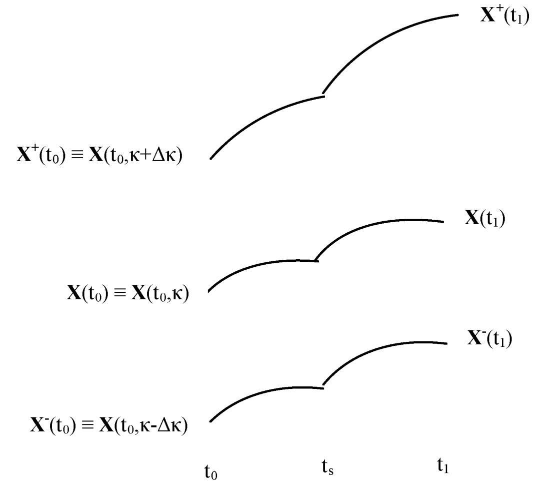

Fig. 1 Finite difference derivatives with discontinuous equations of motion.

$$\frac{\partial \boldsymbol{X}(t_1)}{\partial \kappa} = \frac{\left\{\boldsymbol{X}^+(t_0, \kappa + \Delta\kappa) + \int_{t_0}^{t_1} \dot{\boldsymbol{X}}[\boldsymbol{X}^+(t), t]\,\mathrm{d}t\right\} - \left\{\boldsymbol{X}^-(t_0, \kappa - \Delta\kappa) + \int_{t_0}^{t_1} \dot{\boldsymbol{X}}[\boldsymbol{X}^-(t), t]\,\mathrm{d}t\right\}}{2\Delta\kappa} \tag{1}$$

Error will be introduced into the derivative because, in general, the equations of motion for $\boldsymbol{X}$ must be integrated numerically. In addition to this error source, the choice of $\Delta\kappa$ will introduce error as well. If $\Delta\kappa$ is too large, truncation error will be large; if $\Delta\kappa$ is too small, the derivatives will be corrupted by roundoff error. The optimal value for the perturbation, $\Delta\kappa$, will be different not only for each derivative but for each derivative on each iteration. If the proper perturbation size is used, a good estimate of the accuracy of a derivative of a function with respect to a parameter is one-half the accurate digits to which the function is known.[6] For this case, it would be one-half of the accurate digits to which $\boldsymbol{X}(t_1)$ could be determined. It is important to compute each derivative accurately because the convergence of any parameter optimization algorithm will be limited by the accuracy of the worst derivative.[7]

State Transition Matrix Method

The state transition matrix, which can be determined using the following equations, provides an alternative method to calculate partial derivatives by providing a tool to map perturbations in the spacecraft state from one time to another:

$$\boldsymbol{\Phi}'(t, t_0) = \frac{\partial \boldsymbol{X}'}{\partial \boldsymbol{X}} \boldsymbol{\Phi}(t, t_0) \tag{2}$$

$$\boldsymbol{\Phi}(t_0, t_0) = \boldsymbol{I} \tag{3}$$

The state transition matrix can only map perturbations between two times where the spacecraft state and spacecraft equations of motion are continuous. Figure 2 shows a mapping of a perturbation at time t_0 to the time t_1. The perturbed trajectory is defined to have initial condition $\boldsymbol{X}^+(t_0) = \boldsymbol{X}(t_0, \kappa + \Delta\kappa)$, and the nominal trajectory

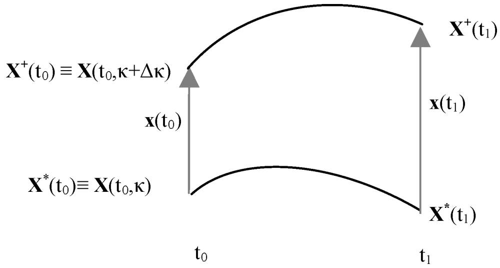

Fig. 2 State transition matrix derivative mappings.

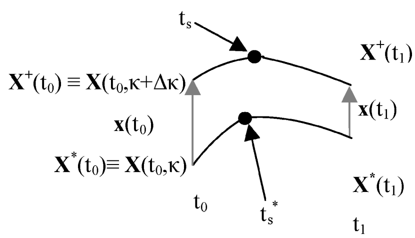

Fig. 3 State transition matrix derivative mappings.

is defined to have initial condition $X^*(t_0) = X(t_0, \kappa)$. Because the derivative of $X(t_0)$ with respect to κ is known, one can write the following equation using a Taylor series:

$$X^+(t_0) - X^*(t_0) \equiv x(t_0) = \Delta\kappa \frac{\partial X(t_0)}{\partial \kappa} + O(\Delta\kappa^2) \tag{4}$$

When the state transition matrix is used, the value of the perturbation at time t_1 can then be written as

$$X^+(t_1) - X^*(t_1) \equiv x(t_1) = \Phi(t_1, t_0)\frac{\partial X(t_0)}{\partial \kappa}\Delta\kappa + O(\Delta\kappa^2) \tag{5}$$

From Eq. (5) and the fundamental definition of a derivative, the following equation is written:

$$\frac{\partial X(t_1)}{\partial \kappa} = \Phi(t_1, t_0)\frac{\partial X(t_0)}{\partial \kappa} \tag{6}$$

Equation (6) requires the equations of motion and the spacecraft state to be continuous between t_0 and t_1 and the partial derivative of the spacecraft state at t_0 with respect to κ to be known analytically. In Ref. 5, a method is provided to overcome discontinuities in the spacecraft state between t_0 and t_1. The mapping of these perturbations is accurate to first order, allowing the calculation of partial derivatives that are error free if the state transition matrix can be computed analytically. In general, the state transition matrix must be determined through numerical integration, which will introduce error into the derivatives calculated using this method. Truncation and roundoff error will not be introduced into the derivatives using the state transition matrix method because these derivatives are taken in the limit as $\Delta\kappa$ goes to zero and no difference is computed as in the finite difference case.

Discontinuities in the Equations of Motion

If there are discontinuities in the equations of motion, modifications must be made to the state transition matrix to map the first-order perturbations in the state. To use this method, the discontinuities in the equations of motion must be finite in number and based on the sign of a switching function that is a function of the spacecraft state and possibly the time. Again, the nominal trajectory is defined to have initial condition $X^*(t_0) = X(t_0, \kappa)$, and the perturbed trajectory has initial condition $X^+(t_0) = X(t_0, \kappa + \Delta\kappa)$. The equations of motion on the nominal trajectory are discontinuous when the switching function $S(X)$ goes through zero at time t_s^*, and the equations of motion on the perturbed trajectory are discontinuous at time t_s as seen in Fig. 3. The equations of motion on the perturbed and nominal trajectories are not discontinuous at the same time because the perturbation will cause the switching function to change sign at a slightly different time as shown in Fig. 4. This time difference is exaggerated significantly in Figs. 3 and 4.

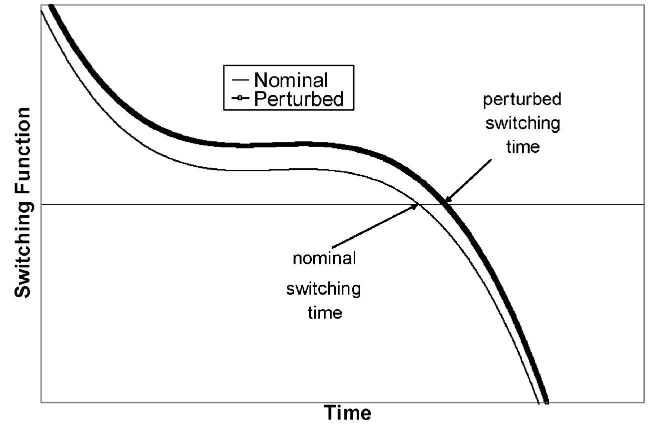

Fig. 4 Change in switching time.

The time-fixed perturbation in the switching function value at the nominal switching time is the difference between the value of the switching function on the perturbed solution and the value of the switching function on the nominal trajectory at the nominal switching time. The time-fixed perturbation can be calculated exactly by using

$$\tilde{\delta} S = \frac{\partial S}{\partial X}\Phi(t_{s-}, t_0)\frac{\partial X(t_0)}{\partial \kappa}\Delta\kappa \tag{7}$$

In Eq. (7), $\Delta\kappa$ is the perturbation to the parameter κ. Multiplying that by the partial derivative of the initial state with respect to κ yields the perturbation in the initial state due to the change in κ at t_0. The state transition matrix between the initial time and just before the switching time maps the perturbation in the state to the nominal switching time. Because the switching function depends only on the state and the time and the time is held fixed, multiplying by the derivative of the switching function with respect to the spacecraft state yields the perturbation in the switching function due to the perturbation in κ.

The fact that the switching function must equal zero when the equations of motion are discontinuous on the perturbed trajectory requires that the time-free perturbation in the switching function must be zero. The time-free perturbation in the switching function is the time-fixed perturbation in the switching function given by Eq. (7) plus the derivative of the switching function with respect to time multiplied by the perturbation in the switching time given by

$$\delta S = \tilde{\delta} S + \left(\frac{\partial S}{\partial X}X'(t_{s-}) + \frac{\partial S}{\partial t}\right)\delta t_s = 0 \tag{8}$$

Equation (8) allows one to determine the first-order change in the switching time because of the perturbation introduced in the state by the change in κ. The change in the switching time is proportional to $\Delta\kappa$.

To determine the derivative of the spacecraft state at some time after the nominal switching time with respect to the parameter κ that affects the spacecraft before the switching time, one must determine both the derivative of the spacecraft immediately before the switching time with respect to κ as well as the change in the spacecraft state because the switching time changes. The first portion of Eq. (9) accounts for the change in the state at t_s that occurs because of the change in the initial state caused by varying κ. The second portion of Eq. (9) accounts for the change that occurs because varying κ changes the time where the equations of motion switch. The derivatives in Eqs. (9) and (10) are mapped to a common time that is necessary to use state transition matrix derivatives. For a discussion of derivatives mapped to a common time, see Ref. 5. Equation (10) demonstrates how to map this forward to a time where the derivative of the spacecraft state with respect to κ is desired. Thus,

$$\left(\frac{\partial X(t_{s+})}{\partial \kappa}\right)_{\&} = \Phi(t_{s-}, t_0)\frac{\partial X(t_0)}{\partial \kappa} + \frac{(X'(t_{s-}) - X'(t_{s+}))\delta t_s}{\Delta\kappa} \tag{9}$$

$$\frac{\partial X(t_1)}{\partial \kappa} = \Phi(t_1, t_{s+})\left(\frac{\partial X(t_{s+})}{\partial \kappa}\right)_{\&} \tag{10}$$

Example Problems

Example Problem 1

Consider a system that begins at $t_0 = 0$ with the initial conditions and equations of motion defined as

$$y(0) = 0, \quad y'(0) = 0, \qquad y'' = \begin{cases} -20 & \text{if} \quad y > 40 \\ 20 & \text{if} \quad y <= 40 \end{cases} \tag{11}$$

The final time t_f is specified to be 3. For this problem y'' will instantaneously change from +20 to −20 at $t = 2$. The partial derivatives of the final state with respect to the initial state are desired. The partial derivatives of the final state with respect to the initial state can be determined analytically using an expansion about the nominal as

$$\frac{\partial y_f}{\partial y_0} = 0, \qquad \frac{\partial y_f}{\partial y_0'} = 1, \qquad \frac{\partial y_f'}{\partial y_0} = -1, \qquad \frac{\partial y_f'}{\partial y_0'} = -1 \tag{12}$$

The state transition matrix for this problem can be determined analytically from t_0 until $t = 2$ and from $t = 2$ until $t = 3$:

$$\Phi(2,0) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \Phi(3,2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{13}$$

When these analytic values for the state transition matrix and Eqs. (7) and (8) are used, the change in the switching time δt when y_0 is increased by $\Delta\kappa$ is $-\Delta\kappa/40$. Similarly, δt when y_0' is increased by $\Delta\kappa$ is $-\Delta\kappa/20$. Using this information in Eq. (10) yields the same values for the partial derivatives as Eq. (12). Because the state transition matrix could be integrated analytically, there is no approximation in the derivatives computed using the state transition matrix method. If the state transition matrix could not be integrated analytically, numerical integration errors would be introduced into the solution.

Example Problem 2

The second example problem is a CSI interplanetary trajectory using intermediate gravity assists to reduce the required propellant. The trajectory consists of a number of segments equal to one plus the number of gravity assists. An example trajectory using one gravity assist is shown in Fig. 5. At the beginning of the first segment, the spacecraft has escaped the departure planet with no excess hyperbolic velocity. The spacecraft then travels on a finite thrust segment until it reaches the first flyby planet. In the example in Fig. 5, the spacecraft thrusts and then coasts during segment one. The flybys are modeled using a zero sphere of influence patched conic approximation discussed in Ref. 5. Following the first flyby, the spacecraft travels on another finite thrust segment, which transports it to either the next gravity assist maneuver or the target planet. In the example in Fig. 5, segment 2 consists of a coasting period, followed by a thrusting period, and a final coasting period. The number of flybys is unlimited and the thrusting structure is free, but the number of

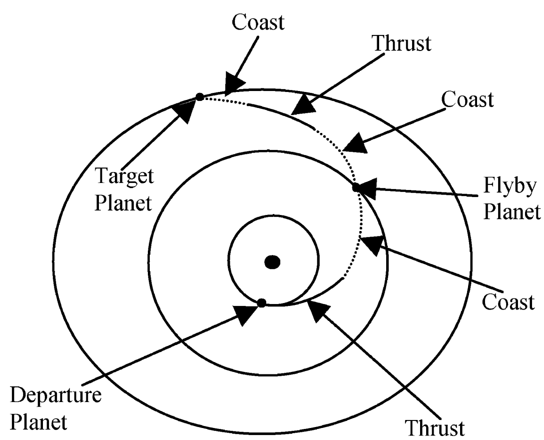

Fig. 5 Finite thrust single-flyby trajectory.

gravity assists and the planet providing each gravity assist must be determined a priori. This method only determines a locally optimal trajectory for a given sequence of gravity assists. The free parameters are the initial time, final time, flyby times, flyby radii, flyby angles, and adjoint control parameters for each segment. The adjoint control parameters are discussed in Ref. 5. The cost function is the total mass used by the engine.

The spacecraft state, $\boldsymbol{X}$, is defined to include the spacecraft position, velocity, and mass as well as its costates as

$$\boldsymbol{X}^T \equiv (\boldsymbol{r}\boldsymbol{v}m\boldsymbol{\lambda}_r \quad \boldsymbol{\lambda}_v \quad \lambda_m)^T \tag{14}$$

The spacecraft engine has a constant specific impulse, and the thrust is constrained to be less than a maximum value. The system Hamiltonian is given as

$$H = \boldsymbol{\lambda}_r^T\boldsymbol{v} + \boldsymbol{\lambda}_v^T\boldsymbol{g}(\boldsymbol{r}) + (T/m)\boldsymbol{\lambda}_v^T\boldsymbol{u} - (T/c)\lambda_m \tag{15}$$

The constraints requiring the spacecraft to intercept the gravity assist planets at the times of the flybys and the target planet at the final time as well as requiring a minimum flyby radius at each gravity assist are adjoined to the cost function by using Lagrange multipliers to form the modified cost function

$$J' = -m_f + \sum_i^q \left\{\boldsymbol{\upsilon}_i^T\left[\boldsymbol{r}\left(t_{\mathrm{fb}_i}\right) - \boldsymbol{r}_{p_i}\left(t_{\mathrm{fb}_i}\right)\right] + \rho_i\left(r_{m_i} - r_{\min_i}\right)\right\} + \boldsymbol{\lambda}^T[\boldsymbol{r}(t_f) - \boldsymbol{r}_t(t_f)] \tag{16}$$

Applying Pontryagin's maximum principle and the first-order necessary conditions for the optimal control problem yields the time derivative of $\boldsymbol{X}$

$$\boldsymbol{X}' \equiv \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{g}(\boldsymbol{r}) + \dfrac{T}{m}\boldsymbol{\lambda}_v \\ -\dfrac{T}{c} \\ -\dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}}\boldsymbol{\lambda}_v \\ -\boldsymbol{\lambda}_r \\ \dfrac{T\lambda_v}{m^2} \end{pmatrix} \tag{17}$$

The thrust must be equal to $T_{\max}$ when the switching function defined as

$$S = \lambda_v/m - \lambda_m/c \tag{18}$$

is greater than zero, and it must be $T_{\min}$ when the switching function is negative. The switching function is only a function of the spacecraft state, and the spacecraft equations of motion will be discontinuous when the switching function crosses through a zero. The state transition matrix can be determined by using Eqs. (2), (3), and

$$\frac{\partial \boldsymbol{X}'}{\partial \boldsymbol{X}} = \begin{pmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times1} \\ \dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}} & \boldsymbol{0}_{3\times3} & -\dfrac{T\boldsymbol{\lambda}_v}{m^2\lambda_v} & \boldsymbol{0}_{3\times3} & \dfrac{T}{m}\left(\dfrac{\boldsymbol{I}_{3\times3}}{\lambda_v} - \dfrac{\boldsymbol{\lambda}_v\boldsymbol{\lambda}_v^T}{\lambda_v^3}\right) & \boldsymbol{0}_{3\times1} \\ \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & 0 & \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & 0 \\ -\dfrac{\partial}{\partial \boldsymbol{r}}\dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}}\boldsymbol{\lambda}_v & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times3} & -\dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}} & \boldsymbol{0}_{3\times1} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times1} & -\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times1} \\ \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & -\dfrac{2T\lambda_v}{m^3} & \boldsymbol{0}_{1\times3} & \dfrac{T\boldsymbol{\lambda}_v}{m^2\lambda_v} & 0 \end{pmatrix} \tag{19}$$

The optimal control problem is solved as a TPBVP with the constraints determined using variational calculus and taking the first variation of Eq. (16). An optimal trajectory must meet the constraints given by the following equations at each flyby[8]:

$$\boldsymbol{r}_{\mathrm{sc}}(t_{\mathrm{fb}}) - \boldsymbol{r}_p(t_{\mathrm{fb}}) = \boldsymbol{0} \tag{20}$$

$$r_m - r_{\min} + \rho^2 = 0 \tag{21}$$

$$\lambda^{\perp}_{v_{\mathrm{fb}-}}\rho = 0 \tag{22}$$

$$\lambda^{\perp}_{v_{\mathrm{fb}-}} = \left|\boldsymbol{\lambda}_{v_{\mathrm{fb}-}} \times \boldsymbol{v}_{\infty_i}\right| = \left|\boldsymbol{\lambda}_{v_{\mathrm{fb}+}} \times \boldsymbol{v}_{\infty_o}\right| = \lambda^{\perp}_{v_{\mathrm{fb}+}} \tag{23}$$

$$\lambda^{\parallel}_{v_{\mathrm{fb}+}} - \lambda^{\parallel}_{v_{\mathrm{fb}-}} = \frac{\lambda^{\perp}_{v_{\mathrm{fb}-}}}{v_{\infty_i}\sin(\delta_{\mathrm{fb}})} \times \left\{2\cos(\delta_{\mathrm{fb}}) + \frac{4r_m \sin^2(\delta_{\mathrm{fb}}/2)\sin(\delta_{\mathrm{fb}})v^2_{\infty_i}}{\mu_{\mathrm{fb}}\left[1-\sin^2(\delta_{\mathrm{fb}}/2)\right]^{\frac{1}{2}}}\right\} \tag{24}$$

$$\boldsymbol{\lambda}_{r_{\mathrm{fb}+}}{}^T\boldsymbol{v}_{\infty_o} - \boldsymbol{\lambda}_{r_{\mathrm{fb}-}}{}^T\boldsymbol{v}_{\infty_i} = \left(\boldsymbol{\lambda}_{v_{\mathrm{fb}+}} - \boldsymbol{\lambda}_{v_{\mathrm{fb}-}}\right)^T\left\{\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}_p(t_{\mathrm{fb}})] - \boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]\right\} \tag{25}$$

$$\left(\boldsymbol{\lambda}^{\perp}_{v_{\mathrm{fb}-}}\right)^T\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_{\infty_o}\right) = 0 \tag{26}$$

$$\left(\boldsymbol{\lambda}^{\perp}_{v_{\mathrm{fb}+}}\right)^T\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_{\infty_o}\right) = 0 \tag{27}$$

In Eq. (25), $\boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]$ is defined to be the gravitational acceleration of the spacecraft at the time of the flyby. Because this is a zero sphere of influence patched conic approximation, the gravitational force due to the flyby planet is not included in $\boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]$. Also note that $\{\mathrm{d}/\mathrm{d}t[\boldsymbol{v}_p(t_{\mathrm{fb}})] - \boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]\}$ would equal zero if the flyby planet had its equations of motion determined by integrating through the same gravity field that the spacecraft is traversing. If, however, the planet being flown by has its position and velocity determined by an ephemeris file that accounts for perturbations not included in the gravity field being used to determine the equations of motion of the spacecraft, this term may not equal zero. In addition, the following equations must be satisfied for an optimal trajectory:

$$\boldsymbol{r}(t_f) - \boldsymbol{r}_t(t_f) = \boldsymbol{0} \tag{28}$$

$$\boldsymbol{\lambda}_v(t_f) = \boldsymbol{0} \tag{29}$$

$$\boldsymbol{\lambda}_r(t_f)^T[\boldsymbol{v}(t_f) - \boldsymbol{v}_t(t_f)] = 0 \tag{30}$$

The free parameters are the initial time, initial adjoint control parameters excluding $\lambda_v(t_0)$ (which is determined from the initial value of the switching function being zero), flyby times, flyby radii, flyby radii slack variables, flyby angles, adjoint control parameters after each flyby, and the final time. There are 7 plus 10 times the number of gravity assists free parameters and an equal number of constraints given by Eqs. (20–30).

Earth–Venus–Saturn Trajectory

An Earth to Saturn intercept trajectory using a CSI engine with a gravity assist provided by Venus is used to demonstrate the benefits of using the state transition matrix to calculate derivatives. The trajectory consists of two segments: the Earth to Venus segment and the Venus to Saturn segment. During each segment the spacecraft can have periods where the thrust is on and periods where the thrust is off. The number of times the engine may turn on or off during any segment is unlimited. The engine is capable of providing a maximum thrust of 20 N and has a specific impulse of 3.2×10^6 s. Clearly this value for the specific impulse is not realistic with current engines. The number was selected to demonstrate the capabilities of the two different derivative methods on long-duration trajectories. The spacecraft has an initial mass of 30,000 kg. The planets' locations and masses are determined using the Jet Propulsion Laboratory DE405 ephemerides.[9] The only gravitational body that affects the spacecraft between gravity assists is the sun. Because the equations of motion are discontinuous, all integration was performed using DLSODAR, a solver for ordinary differential equations that integrates until a root in the switching function occurs.[10] For this trajectory, the minimum flyby radius is set to be zero to simplify the constraint equations. As a result, Eqs. (20–30) can be replaced by

$$\boldsymbol{r}(t_{\mathrm{fb}}) - \boldsymbol{r}_p(t_{\mathrm{fb}}) = \boldsymbol{0} \tag{31}$$

$$\lambda^{\parallel}_{v_{\mathrm{fb}+}} = \lambda^{\parallel}_{v_{\mathrm{fb}-}} \tag{32}$$

$$\boldsymbol{\lambda}_{r_{\mathrm{fb}+}}{}^T\boldsymbol{v}_{\infty_o} - \boldsymbol{\lambda}_{r_{\mathrm{fb}-}}{}^T\boldsymbol{v}_{\infty_i} = \left(\boldsymbol{\lambda}_{v_{\mathrm{fb}+}} - \boldsymbol{\lambda}_{v_{\mathrm{fb}-}}\right)^T\left\{\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{v}_p(t_{\mathrm{fb}})] - \boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]\right\} \tag{33}$$

$$\left(\boldsymbol{\lambda}^{\perp}_{v_{\mathrm{fb}-}}\right)^T\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_{\infty_o}\right) = 0 \tag{34}$$

$$\left(\boldsymbol{\lambda}^{\perp}_{v_{\mathrm{fb}-}}\right)^T\left[\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_{\infty_o}\right) \times \boldsymbol{v}_{\infty_i}\right] = 0 \tag{35}$$

$$\left(\boldsymbol{\lambda}^{\perp}_{v_{\mathrm{fb}+}}\right)^T\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_{\infty_o}\right) = 0 \tag{36}$$

$$\left(\boldsymbol{\lambda}^{\perp}_{v_{\mathrm{fb}+}}\right)^T\left[\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_{\infty_o}\right) \times \boldsymbol{v}_{\infty_i}\right] = 0 \tag{37}$$

$$\boldsymbol{r}(t_f) - \boldsymbol{r}_t(t_f) = \boldsymbol{0} \tag{38}$$

$$\boldsymbol{\lambda}_v(t_f) = \boldsymbol{0} \tag{39}$$

$$\boldsymbol{\lambda}_r(t_f)^T[\boldsymbol{v}(t_f) - \boldsymbol{v}_t(t_f)] = 0 \tag{40}$$

Consequently, the trajectory will be not provide a feasible mission if the flyby radius converges to a value that is less than the radius of Venus. This problem has 16 free parameters: initial time, initial adjoint control parameters excluding $\lambda_v(t_0)$ (which is determined from the initial value of the switching function being zero due to the continuity of the Hamiltonian), flyby time, flyby radius, flyby angle, adjoint control parameters after the flyby, and the final time.

Both state transition matrix derivatives and central difference derivatives were used in a Newton rootfinding method find the optimal values for the free parameters. At each iteration, the new set of free parameters was determined from the previous estimate by using

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \varsigma\left(\left.\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{x}}\right|_{x=x_k}\right)^{-1}\boldsymbol{C}(\boldsymbol{x}_k) \tag{41}$$

The relaxation parameter ζ was set to 10^{-3} when the estimate was far from the solution and increased to 10^{-2} and 10^{-1} as the system converged to the solution. The relaxation parameter could not be set to one using either the central difference or state transition matrix derivatives because it would cause the iterative scheme to overcorrect for errors and diverge from the solution.

The state transition matrix derivatives were able to begin at the initial estimate given in Table 1 and converge to the solution given in Table 1 using 13,000 iterations with ζ set to 10^{-3}, followed by 1000 iterations with ζ set to 10^{-2}, and 200 iterations with ζ set to 10^{-1}. The number of iterations could be reduced by empirically finding the optimal value for ζ on each iteration but that would require more analyst intervention and computational time than using a value of ζ that is not optimal. The locally optimal trajectory shown in Fig. 6 consists of a thrust–coast–thrust–coast–thrust–coast–gravity assist–coast structure. Note the ecliptic plane is used throughout. Figure 7 more clearly shows the trajectory before the gravity assist with the solid lines showing periods when the spacecraft is thrusting and the dotted lines indicating coasting. Figure 7 also clearly shows the discontinuity in the spacecraft velocity at the time of the gravity assist. The switching function can be seen in Fig. 8. The optimal trajectory is infeasible because the optimal value for the flyby radius is 458 km, which is below the surface of Venus. The purpose of this paper

Table 1 Free parameters for Earth–Venus–Saturn trajectory

Parameter[a]	Unit	Initial estimate state transition matrix	Initial estimate central difference	Optimal value
		Free parameters		
t_0	Julian date	2456020	2456038	2456021
t_f	Julian date	2459323	2457474	2458094
α_0	rad	6.4015	6.4106	6.3932
γ_0	rad	−0.1167	−0.1182	−0.1233
α'_0	rad/s	-8.8790×10^{-3}	-8.9950×10^{-3}	-8.7356×10^{-3}
γ'_0	rad/s	3.453610^{-4}	9.673410^{-4}	1.0680×10^{-3}
λ'_{v0}	$AU^{0.5}/day^{2}$[b]	4.3455×10^{-3}	4.5582×10^{-3}	3.9247×10^{-3}
t_{fb1}	Julian date	2456669	2456673	2456669
r_{m1}	km	1290.85	263.71	457.76
β_1	rad	6.6100	6.2368	6.0001
α_1	rad	9.5651	9.7406	9.2750
γ_1	rad	0.1409	−1.4619	−0.2192
α'_1	rad/s	-1.2923×10^{-3}	-5.9511×10^{-3}	-8.7402×10^{-4}
γ'_1	rad/s	-1.9909×10^{-3}	-3.4922×10^{-3}	-2.3075×10^{-2}
λ_{v1}	$AU^{0.5}/day$	0.7612	0.7069	0.6844
λ'_{v1}	$AU^{0.5}/day^2$	3.6671×10^{-3}	2.9146×10^{-2}	-1.3869×10^{-3}
Final mass	kg	29,990.17	29,992.25	29,994.04
		Constraints		
Eq. $(31)_x$	AU	1.1273×10^{-3}	1.6366×10^{-2}	7.8281×10^{-10}
Eq. $(31)_y$	AU	2.6044×10^{-3}	1.3888×10^{-2}	5.1588×10^{-10}
Eq. $(31)_z$	AU	-2.5187×10^{-5}	7.1012×10^{-4}	2.0800×10^{-11}
Eq. (32)	AU/day^2	-3.6553×10^{-2}	-1.8090×10^{-2}	-2.1437×10^{-10}
Eq. (33)	$AU^{1.5}/day^3$	6.2895×10^{-5}	-1.1665×10^{-5}	-7.9756×10^{-13}
Eq. (34)	$AU^{3.5}/day^4$	-4.3168×10^{-8}	1.0486×10^{-7}	3.8812×10^{-16}
Eq. (35)	$AU^{4.5}/day^5$	-3.4440×10^{-10}	-1.5250×10^{-10}	-2.6072×10^{-18}
Eq. (36)	$AU^{3.5}/day^4$	3.7825×10^{-8}	3.5150×10^{-9}	-7.8562×10^{-16}
Eq. (37)	$AU^{4.5}/day^5$	1.1063×10^{-10}	-1.7159×10^{-9}	1.2661×10^{-18}
Eq. $(38)_x$	AU	−0.8942	9.0468×10^{-2}	6.7680×10^{-8}
Eq. $(38)_y$	AU	1.8590	9.3021×10^{-2}	9.3131×10^{-8}
Eq. $(38)_z$	AU	−3.1985	−0.2830	-1.2970×10^{-8}
Eq. $(39)_x$	$AU^{0.5}/day$	0.1619	2.3789×10^{-3}	-1.8433×10^{-8}
Eq. $(39)_y$	$AU^{0.5}/day$	-4.2146×10^{-2}	-2.2087×10^{-2}	-2.6753×10^{-8}
Eq. $(39)_z$	$AU^{0.5}/day$	0.2065	0.1811	4.6338×10^{-8}
Eq. (40)	$AU^{1.5}/day^3$	1.0034×10^{-6}	2.8056×10^{-7}	-1.3207×10^{-13}

[a]Subscripts x, y, and z indicate that the constraint refers to the x, y, or z Cartesian component of the vector constraint.
[b]Astronomical unit.

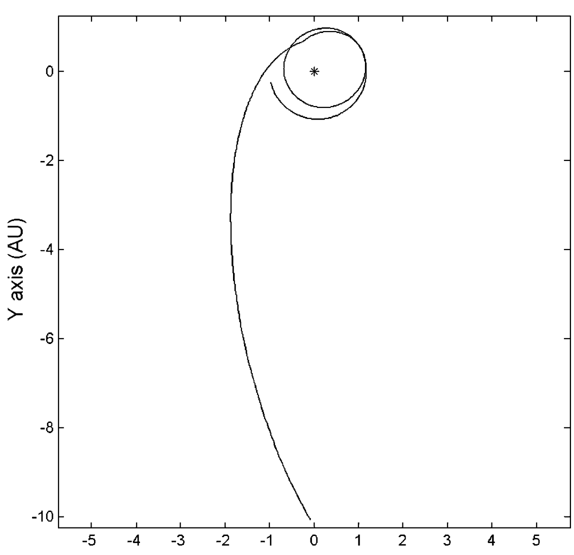

Fig. 6 Optimal Earth–Venus–Saturn trajectory: ——, trajectory and ∗, sun.

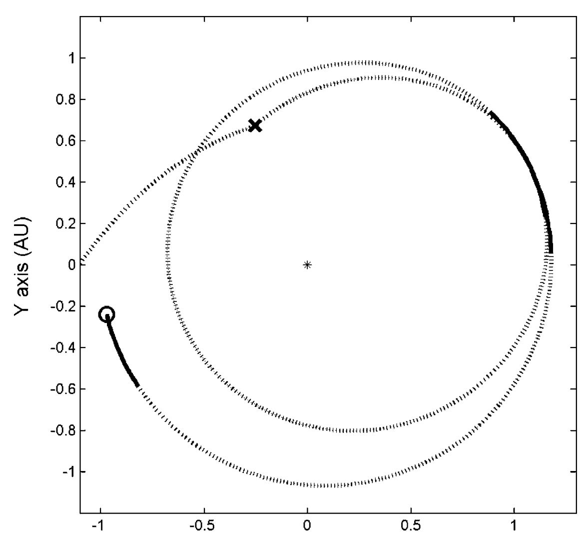

Fig. 7 Optimal Earth–Venus–Saturn trajectory: . . . ., coasting; ——, thrust; ∗, sun; ○, Earth (t_0); and ×, Venus (t_{fb}).

was not to find feasible gravity assist trajectories; it was to demonstrate whether state transition matrix derivatives can be beneficial to optimization algorithms when the spacecraft state and equations of motion are discontinuous. An optimal feasible solution can be found by employing Eqs. (20–30) instead of Eqs. (31–40) because Eqs. (20–30) are capable of enforcing a minimum flyby radius.

When the same initial guess is used to start and central difference derivatives tuned by using Hull's method[7] instead of the state transition matrix derivatives are used, the Newton iteration method is unable to converge to the optimal values. Instead it diverges after approximately 5000 iterations with ζ set to 10^{-3} without approaching the optimal solution found by the state transition matrix method. An attempt was made to determine if the central difference derivatives would converge to the optimal solution from an initial estimate that was closer to the optimal solution than the initial estimate used by the state transition matrix derivative method. To obtain an initial estimate that was closer to the optimal solution, the state transition matrix derivatives were employed for a multiple of 1000 iterations with the relaxation parameter set to 10^{-3}. After a multiple of 1000 iterations with the state transition matrix derivatives, the process was continued using central difference derivatives. The central difference derivatives diverged when 10,000 or less state transition matrix derivative iterations were used and converged when 11,000 or greater state transitions matrix derivative iterations were used before beginning to use central difference derivatives. Table 1 shows the values of the free parameters when central difference derivatives began to be used to successfully determine the optimal values of the free parameters.

Example Problem 3

The third example problem is the same as the second example problem except for two modifications. First, there is no gravity assist in this problem. Second, the solution method does not employ variational calculus to determine constraints and form a TPBVP problem. Instead, the kinematic boundary conditions are satisfied through the use of equality constraints and the transversality conditions are satisfied by minimizing the cost function using a sequential quadratic programming code.[11] This method requires one to determine the derivatives of the cost and constraint functions with respect to all of the free parameters. The accuracy with which the optimal trajectory can be found depends on the accuracy of these derivatives. The Euler–Lagrange equations are numerically integrated and the continuous control still satisfies the Pontryagin maximum principle. The spacecraft is constrained to intercept the target planet at the final time, and the cost function is the total mass consumed.

To test the possible benefits of the state transition matrix method derivatives, a trajectory using a CSI engine to transport the spacecraft from Earth to Saturn is sought. The engine parameters are the same as in example 2. Applying Pontryagin's maximum principle again yields Eqs. (14) and (17) for the spacecraft state and its time derivative. The free parameters for this case are the initial time, final time, and the initial adjoint control parameters. Although the sequential quadratic programming code only directly enforced Eq. (38), an optimal trajectory must satisfy Eqs. (39) and (40) as well. Both state transition matrix derivatives and central difference derivatives were used to calculate an optimal trajectory starting from the same initial guess listed in Table 2 using the sequential quadratic programming code. Table 2 also shows the solution that both derivative methods converged to as well as the optimal solution. The optimal solution was determined by using the rootfinding method described in example problem 2. The state transition matrix method derivatives were able to converge to nearly the optimal solution. The central difference derivative solution is extremely far from the optimal solution. In fact, the sequential quadratic programming code found the central difference solution using both central difference derivatives and state transition matrix derivatives. When using state transition matrix

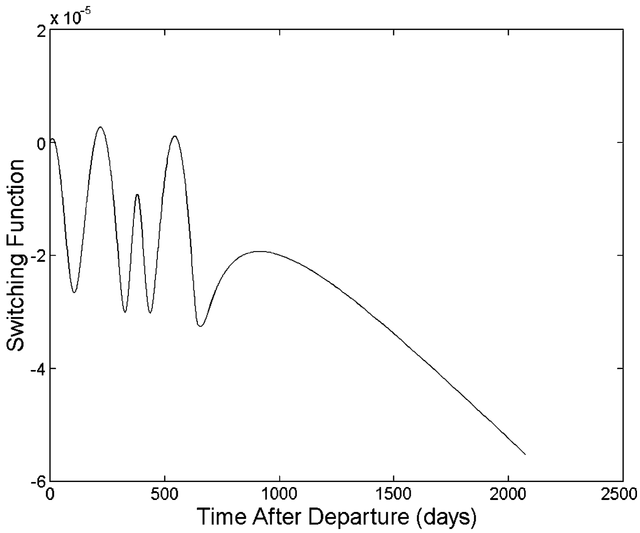

Fig. 8 Switching structure for optimal Earth–Venus–Saturn trajectory.

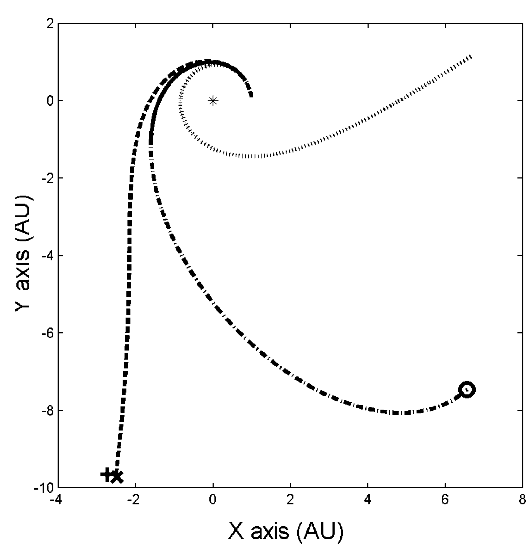

Fig. 9 Earth–Saturn trajectories: . . . ., initial estimate; – – –, CD (thrust); ——, STM (thrust); –·–·, STM (coast); ∗, sun; ○, Saturn (t_f) STM; ×, Saturn (t_f) CD; and +, Saturn (t_f) estimate.

Table 2 Free parameters for Earth–Saturn trajectory

Parameter[a]	Unit	Initial estimate	Converged values central difference	Converged values state transition matrix	Optimal value
		Free parameters			
t_0	Julian date	2456952.78	2456925.77	2456965.48	2456964.83
t_f	Julian date	2457589.79	2457636.91	2459477.13	2459455.70
α_0	rad	6.2981	3.4966	3.4193	3.3659
γ_0	rad	2.2007	3.5252	3.0690	3.0808
α_0'	rad/s	-7.9599×10^{-2}	2.4090×10^{-2}	$-3.3591E \times 10^{-3}$	-2.3463×10^{-3}
γ_0'	rad/s	1.4611×10^{-2}	-1.2437×10^{-2}	-3.6910×10^{-5}	4.7658×10^{-4}
λ_{v0}	$AU^{0.5}/day^2$	7.9702×10^{-2}	2.56780×10^{-2}	1.5106×10^{-2}	1.2577×10^{-2}
Final mass	kg	29,964.94	29,960.85	29,988.72	29,988.73
		Constraints			
Eq. $(38)_x$	AU	9.3835	-3.5578×10^{-2}	8.2694×10^{-8}	2.6636×10^{-9}
Eq. $(38)_y$	AU	10.8002	7.9357×10^{-3}	-6.2508×10^{-6}	-5.9794×10^{-11}
Eq. $(38)_z$	AU	−0.9918	1.1228×10^{-3}	-1.4807×10^{-7}	-3.3001×10^{-11}
Eq. $(39)_x$	$AU^{0.5}/day$	30.5139	−8.2630	−0.9056	-3.2940×10^{-9}
Eq. $(39)_y$	$AU^{0.5}/day$	25.6998	−15.1167	9.6117×10^{-2}	2.0821×10^{-8}
Eq. $(39)_z$	$AU^{0.5}/day$	−3.8537	−0.8742	−0.4560	3.5364×10^{-10}
Eq. (40)	$AU^{1.5}/day^3$	5.5250×10^{-5}	-5.1838×10^{-4}	4.9470×10^{-7}	1.1447×10^{-14}

[a]Subscripts x, y, and z indicate that the constraint refers to the x, y, or z Cartesian component of the vector constraint.

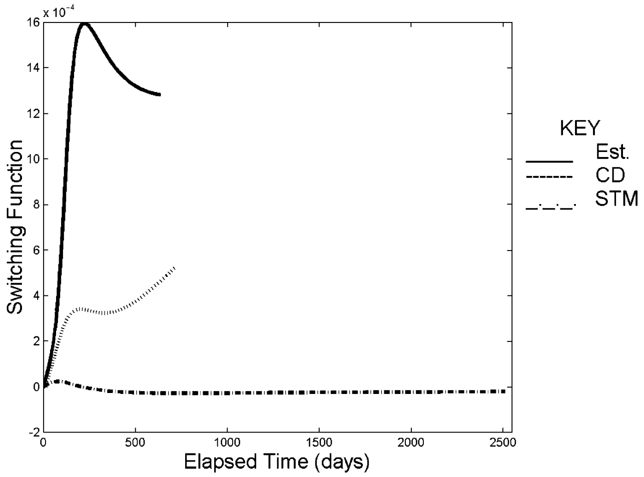

Fig. 10 Switching structure for Earth–Saturn trajectories.

derivatives, the sequential quadratic programming code was able to continue past this solution in search of a better one. The central difference derivatives were not accurate enough for the sequential quadratic programming code to move past what should have been an intermediate nonoptimal solution.

Figure 9 shows the trajectory computed using the initial estimate of the free parameters, as well as the solutions calculated using central difference derivatives and state transition matrix derivatives. The largest difference between the solution found using the state transition matrix derivatives and the solution found using central difference (CD) derivatives is the state transition matrix solution consists of a thrusting segment followed by a coasting segment, whereas the central difference solution is one continuous thrusting segment. The switching function for all three of the trajectories is shown in Fig. 10. Because the state transition matrix (STM) solution's switching function is positive for a much shorter time than the other two switching functions, the spacecraft thrusts for a shorter period of time on the state transition trajectory. Consequently, this trajectory consumes less mass.

Conclusions

The STM method provides a tool to calculate the derivatives necessary to calculate optimal interplanetary trajectories even when the equations of motion and the spacecraft state are discontinuous. The partial derivative of the spacecraft state after the discontinuity with respect to the spacecraft state before the discontinuity must be known analytically. When both sequential quadratic programming algorithms and rootfinding algorithms are used, the state transition matrix derivatives are shown to converge to the optimal solution from initial estimates of the free parameters where CD derivatives fail to converge. The main disadvantage of using the STM to calculate derivatives is that it requires numerous analytic partial derivatives to be determined and coded. Whether the STM method or CD method requires less total time (time required to derive conditions and move from initial estimates to final solution) depends on the number of solutions sought and the quality of the initial estimates for the free parameters.

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